INDIAN STATISTICAL INSTITUTE

Probability Theory II: B. Math (Hons.) I

Semester II, Academic Year 2018-19

Final Exam (Total Marks: 50)

Date: 02/05/2019 **Time: 10:00 am - 1:00 pm**

- **Please write your name on top of your answer-script.**
- **Show all your works and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. A continuous random vector (X, Y) has a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} 2/3 & \text{if } x > 0, y > 0, x + y < 1, \\ c & \text{if } x < 1, y < 1, x + y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) (2 marks) Find c.

   (b) (4 marks) Compute the conditional probability density function of Y given X.

   (c) (4 marks) Calculate $E(Y|X)$.

2. (10 marks) Suppose $(X_1, X_2, X_3, X_4, X_5) \sim D(\alpha_1, \alpha_2, \alpha_3, \alpha_4, ; \beta)$ (the notation is as used in the class), where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta$ are positive parameters. Compute, with full justification, a joint probability density function of $(X_2 + X_4, X_5, X_3)$.

3. (10 marks) Suppose that X is a random variable with finite mean and $\varphi(t)$ is its characteristic function. Show that $\varphi(t)$ is differentiable and with proper justification, express $E(X)$ in terms of the derivative of $\varphi(t)$.

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4. (10 marks) Fix a positive integer $n > 1$. Let $X_1, X_2, \ldots, X_n$ be i.i.d. standard normal random variables. For $k = 1, \ldots, n-1$, define

$$Y_k = \frac{1}{\sqrt{k(k+1)}} \left(\sum_{i=1}^{k} X_i - k X_{k+1} \right).$$

Then show that $Y_1, \ldots, Y_{n-1}$ are also i.i.d. standard normal random variables.

5. (10 marks) Suppose it is given that a sequence of random variables $\{X_n\}$ converges in distribution. Show that for every $\epsilon \in (0,1)$, there exist $a, b \in \mathbb{R}$ such that

$$P\big(X_n \in (a, b]\big) \geq 1 - \epsilon$$

for all $n \geq 1$.